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Approximation of Continuous Functions in p-Adic Analysis

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1. INTRODUCTION

Let Q_p be the field of p-adic numbers with a p-adic valuation $|\cdot|_p$ that has the properties

$$|x+y|_p \leqslant \max(|x|_p, |y|_p)$$
$$|xy|_p = |x|_p |y|_p$$

and is so normalized that $|p^n|_p = p^{-n}$. As is well-known, Q_p is a complete metric space with respect to the distance function $d(x, y) = |x - y|_p$ and the field of rational numbers Q is dense in Q_p.

We will consider functions defined on the set of p-adic integers

$$I = \{x : |x|_p \leqslant 1\},$$

with values in Q_p. The theory of p-adic valued functions in the period from the introduction of p-adic numbers by Hensel [1] at the end of the nineteenth century until very recent times resembles closely the theory of analytic functions.

The study of p-adic valued functions from the point of view of the constructive theory of functions and approximation theory was initiated in 1944 by Dieudonné [2], who proved that every continuous p-adic valued function on a compact subset of Q_p can be approximated uniformly by polynomials. A more constructive proof of this result was given in 1958 by Mahler [3, 4] (see also [5, Chap. 6]) for continuous functions on I. Mahler's theorem can be stated as follows.

THEOREM A. *Let $f\colon I \to Q_p$ be a continuous function and let*

$$a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k), \qquad k = 0, 1, 2,.... \tag{1.1}$$

Then the series

$$\sum_{k=0}^{\infty} a_k(f)\binom{x}{k} \tag{1.2}$$

converges uniformly on I and

$$f(x) = \sum_{k=0}^{\infty} a_k(f)\binom{x}{k} \text{ for every } x \in I. \tag{1.3}$$

Mahler's series (1.2) is clearly the analogue of Newton's interpolatory series and it was natural to use this series since a continuous function f on I is completely determined by its values on the set $J = \{0, 1, 2,...\}$, which is dense in I. However, the remarkable fact here is that series (1.2) converges uniformly for every continuous function f on I. This is clearly equivalent to stating that for every continuous f on I

$$\lim_{n\to\infty} | a_n(f)|_p = 0. \tag{1.4}$$

A simple analytic proof of this result was given recently in [6].

Another, even more remarkable property of Mahler's expansion is that the nth partial sum of series (1.2) is a polynomial of best approximation of degree $\leqslant n$ to f on I. This result can be established quite easily in p-adic analysis.

If f is a continuous function on I, and $a_n(f)$ is defined by (1.1), then

$$| a_n(f)|_p \leqslant \max_{x\in I} | f(x)|_p \qquad \text{for every } n = 0, 1, 2,.... \tag{1.5}$$

On the other hand, since

$$\left|\binom{x}{k}\right|_p \leqslant 1 \qquad \text{for } x \in I,$$

from (1.3) it follows that

$$| f(x)|_p \leqslant \max_{k\in J} | a_k(f)|_p . \tag{1.6}$$

From inequalities (1.5) and (1.6) it follows that

$$\max_{x\in I} | f(x)|_p = \max_{k\in J} | a_k(f)|_p . \tag{1.7}$$

Since any polynomial P of degree $\leqslant n$ is of the form

$$P(x) = \sum_{k=0}^{n} \alpha_k \binom{x}{k},$$

we have, by (1.7),

$$\max_{x \in I} | f(x) - P(x)|_p = \max (| a_k(f) - \alpha_k |_p , 1 \leqslant k \leqslant n; \max_{k \geqslant n+1} | a_k(f)|_p) \geqslant \max_{k \geqslant n+1} | a_k(f)|_p .$$

On the other hand, if

$$P_n^*(x) = \sum_{k=0}^{n} a_k(f)\binom{x}{k}$$

is the nth partial sum of Mahler's expansion (1.3) of f, we have, by (1.7),

$$\max_{x \in I} | f(x) - P_n^*(x)|_p = \max_{k \geqslant n+1} | a_k(f)|_p .$$

Hence, if we denote by $\mathbb{P}_n$ the set of all polynomials of degree $\leqslant n$, we have

$$\max_{x \in I} | f(x) - P_n^*(x)|_p = \inf_{P \in \mathbb{P}_n} (\max_{x \in I} | f(x) - P(x)|_p) = \max_{k \geqslant n+1} | a_k(f)|_p .$$

This fact that the nth partial sum of Mahler's expansion (1.3) of a continuous function f on I is a polynomial of best approximation to f of degree $\leqslant n$ seems to be quite important, even if the polynomials of best approximation are not unique. It indicates that one should expect that the structural properties of a continuous function f could be characterized, as in real analysis, in terms of the asymptotic properties of the coefficients $a_n(f)$.

The aim of this paper is to present several results of this type. These results are stated in Section 2. Section 3 contains lemmas necessary for the proof of our theorems and, finally, Section 4 contains proofs of the theorems.

2. Results

The first problem that will be considered here is closely related to Mahler's Theorem A and it can be stated as follows.

Let f be a p-adic valued function defined on $I = \{x : | x |_p \leqslant 1\}$ and let the p-adic numbers $(a_n(f))$ be defined on $J = \{0, 1, 2,...\}$ by

$$a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k), \qquad n = 0, 1, 2,... \tag{2.1}$$

What are the necessary and sufficient conditions for f in order that

$$\lim_{n \to \infty} | a_n(f)|_p = 0 ? \tag{2.2}$$

In view of Mahler's Theorem A, the continuity of f on I is clearly a sufficient condition for (2.2). However, since $a_n(f)$ is defined in terms of the values of

f on J, the continuity of f is certainly not a necessary condition for (2.2). We will show here first that a weaker condition, which can be described as the uniform continuity of f on J, is both necessary and sufficient for (2.2). This result can be stated more precisely as follows:

THEOREM 1. *Let f be a p-adic valued function on the set $J = \{0, 1, 2,...\}$ and let $(a_n(f))$ be the sequence of p-adic numbers defined by* (2.1). *We have then*

$$\lim_{n\to\infty} | a_n(f)|_p = 0$$

if and only if

$$w_f(p^{-t}) = \max_{k\in J} | f(k + p^t) - f(k)|_p \to 0 \ (t \to \infty). \tag{2.3}$$

Condition (2.3) is clearly satisfied if f is a continuous p-adic valued function on I.

Next, we consider functions that satisfy continuity conditions stronger than (2.3). For such functions it is natural to expect more precise results than (2.2).

THEOREM 2. *Let f be a p-adic valued function defined on the set $J = \{0, 1, 2,...\}$, let $(a_n(f))$ be the sequence of p-adic numbers defined by* (2.1) *and let $0 < \alpha \leqslant 1$. We have then*

$$| a_n(f)|_p = \mathcal{O}(n^{-\alpha}) \qquad (n \to \infty) \tag{2.4}$$

if and only if

$$w_f(p^{-t}) = \max_{k\in J} | f(k + p^t) - f(k)|_p = \mathcal{O}(p^{-\alpha t}) \qquad (t \to \infty). \tag{2.5}$$

A special case of Theorem 2 corresponding to $\alpha = 1$, in a slightly modified form, was suggested as a research problem by Prof. Mahler in his lectures on p-adic analysis at the Ohio State University in the summer quarter 1973.

A class of p-adic valued functions which satisfy both conditions (2.3) and (2.5) are p^t-periodic functions. A function $f\colon I \to Q_p$ is p^t-periodic ($t \geqslant 1$) if

$$f(x + p^t) = f(x) \qquad \text{for every } x \in I.$$

For p^t-periodic functions on J we have the following much stronger result than (2.4).

THEOREM 3. *Let $f\colon J \to Q_p$ be a p^t-periodic function with $t \geqslant 1$ and let $(a_n(f))$ be the sequence of p-adic numbers defined by* (2.1). *Then for every $n \geqslant p^t$*

$$| a_n(f)|_p \leqslant p^{-[n/p^t]} \max_{0\leqslant k\leqslant p^t-1} | a_k(f)|_p . \tag{2.6}$$

If, in particular, $(f(n))$ is a p^t-periodic sequence of rational integers, Theorem 3 states that, for every $n \geqslant p^t$, the integer

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k)$$

is divisible by $p^{[n/p^t]}$.

To study more general problems of this type we will choose an arbitrary continuous, nondecreasing function Ω on [0, 1], with $\Omega(0) = 0$, and we will consider p-adic valued functions on I that satisfy the condition

$$w_f(p^{-t}) = \max_{k \in J} | f(k + p^t) - f(k)|_p = \mathcal{O}(\Omega(p^{-t})) \qquad (t \to \infty). \quad (2.7)$$

In view of Theorem 2, it would be natural to expect the estimate

$$| a_n(f)|_p = \mathcal{O}(\Omega(1/n)) \qquad (n \to \infty), \quad (2.8)$$

and, conversely, that (2.8) should imply (2.7). However, results of this generality seem to be very difficult to prove, if they are correct at all. We are able to show that (2.7) implies (2.8) if the speed with which Ω converges to zero is restricted by the condition

$$\liminf_{\lambda \to 0+} \frac{\Omega(\lambda/p)}{\Omega(\lambda)} > 0. \quad (2.9)$$

As examples of function satisfying this condition we mention in particular

$$\Omega(\lambda) = (\log (1/\lambda)^{-\alpha} \qquad (0 < \alpha < \infty) \quad (2.10)$$

$$\Omega(\lambda) = \lambda^\alpha \qquad (0 < \alpha < \infty) \quad (2.11)$$

and

$$\Omega(\lambda) = \exp(-c \log^\alpha(1/\lambda)) \qquad (0 < \alpha < 1), \quad (2.12)$$

but the function $\Omega(\lambda) = \exp(-1/\lambda)$ does not satisfy condition (2.9).

THEOREM 4. *Let f be a p-adic valued function on J such that*

$$w_f(p^{-t}) = \max_{k \in J} | f(k + p^t) - f(k)|_p = \mathcal{O}(\Omega(p^{-t})) \qquad (t \to \infty).$$

and let $(a_n(f))$ be the sequence of p-adic numbers defined by (2.1). *If the function Ω satisfies condition* (2.9), *then*

$$| a_n(f)|_p = \mathcal{O}(\Omega(1/n)) \qquad (n \to \infty).$$

To prove a converse statement we need an even more restrictive hypothesis about Ω. We shall have to assume that there exists a subinterval $(0, \delta)$ of $(0, 1)$ such that

$$\Omega(\lambda/p)/(\Omega(\lambda)) \geqslant 1/p \qquad \text{for every} \quad \lambda \in (0, \delta). \tag{2.13}$$

This condition is satisfied if Ω is defined as in (2.10) and (2.12). It is also satisfied if $\Omega(\lambda) = \lambda^\alpha$ with $0 < \alpha \leqslant 1$, but it is clearly not satisfied if $\alpha > 1$. More generally, (2.13) is satisfied whenever

$$\liminf_{\lambda\to 0+} \frac{\Omega(\lambda/p)}{\Omega(\lambda)} > \frac{1}{p}.$$

THEOREM 5. *Let f be a p-adic valued function on J and let $(a_n(f))$ be the sequence of p-adic numbers defined by* (2.1). *If the function Ω satisfies condition* (2.13) *and if*

$$|a_n(f)|_p = \mathcal{O}(\Omega(1/n)) \qquad (n \to \infty),$$

then

$$w_f(p^{-t}) = \max_{k\in J} |f(k + p^t) - f(k)|_p = \mathcal{O}(\Omega(p^{-t})) \qquad (t \to \infty).$$

3. LEMMAS

For the proof of Theorems 1–5 we need a number of preliminary results. We will always assume here that f is a p-adic valued function defined on the set $J = \{0, 1, 2,...\}$ and that

$$a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k), \qquad n = 0, 1, 2,.... \tag{3.1}$$

We also write

$$w_f(p^{-t}) = \max_{k\in J} |f(k + p^t) - f(k)|_p .$$

LEMMA 1. *If*

$$w_f(p^{-t}) \to 0 \qquad (t \to \infty) \tag{3.2}$$

then f is bounded on J.

Proof of Lemma 1. Choose T such that

$$M_T(f) = \max_{0\leqslant k\leqslant T} |f(k)|_p >$$

By (3.2) we can find a t such that

$$\max_{k\in J} |f(k+p^t)-f(k)|_p \leqslant M_T(f). \tag{3.3}$$

We will show that

$$|f(n)|_p \leqslant M_{T+p^t}(f) \qquad \text{for every } n \in J. \tag{3.4}$$

This inequality is clearly satisfied if $0 \leqslant n \leqslant p^t$. Suppose therefore that $n \geqslant p^t$. Then $sp^t \leqslant n < (s+1)\,p^t$ for some $s \geqslant 1$ and so

$$f(n) = f(n-sp^t) + \sum_{j=1}^{s} (f(n-jp^t+p^t) - f(n-jp^t)).$$

Since $0 \leqslant n - sp^t \leqslant p^t$, we have, by (3.3),

$$|f(n)|_p \leqslant \max(M_{p^t}(f), M_T(f)) \leqslant M_{p^t+T}(f).$$

Hence (3.4) holds also if $n \geqslant p^t$ and the lemma is proved.

LEMMA 2. *For every $n \geqslant 0$ and $m \geqslant 1$ we have*

$$\sum_{j=1}^{m} \binom{m}{j} a_{n+j}(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}(f(k+m)-f(k)) \tag{3.5}$$

and

$$f(n+m)-f(n) = \sum_{k=0}^{n} \left(\sum_{j=1}^{m} a_{k+j}(f)\binom{m}{j}\right)\binom{n}{k}. \tag{3.6}$$

Proof of Lemma 2. Let $\tau_m f(x) = f(x+m)$. The proof of relations (3.5) and (3.6) consists essentially in expressing the coefficients of the translated function $\tau_m f$ in terms of the coefficients of f. Let

$$a_n(\tau_m f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \tau_m f(k). \tag{3.7}$$

Then

$$\tau_m f(n) = \sum_{k=0}^{n} a_k(\tau_m f)\binom{n}{k}. \tag{3.8}$$

In [6] it was proved that

$$a_n(\tau_m f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(m+k) = \sum_{j=0}^{m} \binom{m}{j} a_{n+j}(f). \tag{3.9}$$

Relation (3.5) is clearly a consequence of (3.9) and (3.6) follows from (3.8) and (3.9).

LEMMA 3. *For every $n \geqslant 0$ and $t \geqslant 1$ we have*

$$| a_{n+p^t}(f)|_p \leqslant \max ((1/p)| a_{n+j} |_p , 1 \leqslant j \leqslant p^t - 1 ; w_f(p^{-t})). \tag{3.10}$$

Proof of Lemma 3. By (3.5), with $m = p^t$, we have

$$a_{n+p^t}(f) = - \sum_{j=1}^{p^t-1} \binom{p^t}{j} a_{n+j}(f) + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}(f(k + p^t) - f(k))$$

and (3.10) follows since $p \mid \binom{p^t}{j}$ for every $j = 1,..., p^t - 1$.

While Lemma 3 is sufficient for the proof of Theorems 1–3, we need a more refined version of that lemma for the proof of Theorem 4.

LEMMA 4. *Let $f: J \to Q_p$ be a bounded function and let*

$$\Lambda(n) = \max_{k \geqslant n} | a_k(f)|_p .$$

Then, for every integer $\sigma \geqslant 1$ and $t \geqslant 1$ we have

$$\Lambda((\sigma + 1) p^t) \leqslant \max((1/p^\sigma) \Lambda(p^t), w_f(p^{-t})).$$

Proof of Lemma 4. By Lemma 3 we have, for every $t \geqslant 1$ and $n \geqslant 0$,

$$| a_{n+p^t}(f)|_p \leqslant \max ((1/p)| a_{n+j}(f)|_p , 1 \leqslant j \leqslant p^t - 1 ; w_f(p^{-t})).$$

Replacing here n by $n + sp^t$ we find that

$$\begin{aligned} | a_{n+(s+1)p^t}(f)|_p &\leqslant \max ((1/p)| a_{n+sp^t+j}(f)|_p , 1 \leqslant j \leqslant p^t - 1 ; w_f(p^{-t})) \\ &\leqslant \max ((1/p) \Lambda(sp^t), w_f(p^{-t})). \end{aligned}$$

Since this inequality holds for every $n \geqslant 0$, it follows that

$$\Lambda((s + 1) p^t) \leqslant \max((1/p) \Lambda(sp^t), w_f(p^{-t})). \tag{3.11}$$

In particular, we have

$$\Lambda(2p^t) \leqslant \max((1/p) \Lambda(p^t), w_f(p^{-t})).$$

Hence, Lemma 4 is true if $\sigma = 1$.

Now, we show by induction that Lemma 4 is true for every integer $\sigma \geqslant 1$. Suppose that the lemma is true for $\sigma = s$. We have then, by induction hypothesis,

$$\Lambda((s + 1) p^t) \leqslant \max((1/p^s) \Lambda(p^t), w_f(p^{-t})). \tag{3.12}$$

By (3.11) and (3.12) we have then

$$\begin{aligned}\Lambda((s+2)\,p^t) &\leqslant \max((1/p)\,\Lambda((s+1)\,p^t),\, w_f(p^{-t}))\\ &\leqslant \max((1/p)\max((1/p^s)\,\Lambda(p^t),\, w_f(p^{-t})),\, w_f(p^{-t}))\\ &\leqslant \max((1/p^{s+1})\,\Lambda(p^t),\, w_f(p^{-t}))\end{aligned}$$

and the lemma is proved.

LEMMA 5. *Let Ω be a continuous and nondecreasing function on* [0, 1] *with* $\Omega(0) = 0$. *If there exists* $\delta \in (0, 1)$ *such that*

$$\Omega(\lambda/p) \geqslant p^{-1}\Omega(\lambda) \qquad \textit{for every} \quad \lambda \in (0, \delta), \tag{3.13}$$

then, as $t \to \infty$,

$$\max_{1 \leqslant j \leqslant p^t} j\Omega(1/j) = \mathcal{O}(p^t\Omega(p^{-t})).$$

Proof of Lemma 5. Let T be a fixed integer such that $1/p^{T-1} < \delta$ and $t > T$. We have then

$$\max_{1 \leqslant j \leqslant p^t} j\Omega(1/j) \leqslant \max_{1 \leqslant j \leqslant p^T} j\Omega(1/j) + \max_{p^T \leqslant j \leqslant p^t} j\Omega(1/j). \tag{3.14}$$

The first term on the right-hand side of (3.14) can be estimated as follows. From (3.13) it follows that for every $\lambda \in (0, \delta)$ and $s = 0, 1, 2,\ldots$ we have

$$\Omega(\lambda/p^{s+1}) \geqslant p^{-s-1}\Omega(\lambda). \tag{3.15}$$

By choosing $\lambda = \delta/2$ we obtain, in particular, the inequality

$$\begin{aligned}\Omega(p^{-t}) &\geqslant \Omega(\delta/2p^{t+1}) \geqslant p^{-t-1}\Omega(\delta/2)\\ \text{or} \qquad 1 &\leqslant (p/\Omega(\delta/2))\,p^t\Omega(p^{-t}).\end{aligned} \tag{3.16}$$

Next, suppose that $p^T \leqslant j \leqslant p^t$. Then $p^{r-1} \leqslant j \leqslant p^r$ for some integer r such that $T + 1 \leqslant r \leqslant t$. Hence

$$j\Omega(1/j) \leqslant p^r\Omega(p^{-r+1}).$$

Since $1/p^{r-1} \leqslant 1/p^{T-1} < \delta$, from (3.15), with $\lambda = 1/p^{r-1}$ and $s = t - r$, it follows that

$$\Omega(p^{-r+1}) \leqslant p^{t-r+1}\Omega(p^{-r+1}/p^{t-r+1}) \leqslant p^{t-r+1}\Omega(p^{-t}).$$

Hence, for every j such that $p^T \leqslant j \leqslant p^t$, we have

$$j\Omega(1/j) \leqslant p^{t+1}\Omega(p^{-t})$$

and it follows that

$$\max_{p^T \leqslant j \leqslant p^t} j\Omega(1/j) \leqslant p^{t+1}\Omega(p^{-t}). \tag{3.17}$$

Finally, from (3.14), (3.16) and (3.17) follows the statement of Lemma 5.

4. Proofs of the Theorems

Proof of Theorem 1. Sufficiency of (2.3). Since this part of Theorem 1 was essentially proved in [6], we shall give here only a brief outline of the proof. Given $s > 0$, choose t such that

$$w_f(p^{-t}) = \max_{k \in J} |f(k + p^t) - f(k)|_p \leqslant p^{-s}.$$

By (3.10) we have

$$|a_{n+p^t}(f)|_p \leqslant \max((1/p)|a_{n+j}|_p, 1 \leqslant j \leqslant p^t - 1; (1/p^s)). \tag{4.1}$$

Since, by Lemma 1, f is a bounded function on J, we can without loss of generality assume that $|f(k)|_p \leqslant 1$ for every $k \in J$. From (2.1) follows then that $|a_n(f)|_p \leqslant 1$ for all $n \geqslant 0$. Using this inequality and (4.1) we find that $|a_n(f)|_p \leqslant 1/p$ for all $n \geqslant p^t$. Continuing this process we find that

$$|a_n(f)|_p \leqslant 1/p^s \quad \text{for} \quad n \geqslant sp^t$$

and (2.2) follows since s can be chosen arbitrarily large.

Necessity of (2.3). Suppose next that (2.2) holds. Then

$$\max_{k \in J} |a_k(f)|_p < \infty$$

and

$$A(n) = \max_{k \geqslant n} |a_k(f)|_p \to 0 \qquad (n \to \infty). \tag{4.2}$$

By (3.6) we have

$$f(n + p^t) - f(n) = \sum_{k=0}^{n} \left(\sum_{j=1}^{p^t} a_{k+j}(f) \binom{p^t}{j} \right) \binom{n}{k}$$

and so

$$|f(n + p^t) - f(n)|_p \leqslant \max_{k \in J} \left| \sum_{j=1}^{p^t} a_{k+j}(f) \binom{p^t}{j} \right|_p. \tag{4.3}$$

Let N be a fixed integer and $p^t \geqslant N$. We have then

$$\sum_{j=1}^{p^t} a_{k+j}(f)\binom{p^t}{j} = p^t \sum_{j=1}^{N} \frac{a_{k+j}(f)}{p^t}\binom{p^t}{j} + \sum_{j=N+1}^{p^t} a_{k+j}(f)\binom{p^t}{j}$$

$$= p^t \sum_{j=1}^{N} \frac{a_{k+j}(f)}{j}\binom{p^t-1}{j-1} + \sum_{j=N+1}^{p^t} a_{k+j}(f)\binom{p^t}{j}.$$

Hence

$$\left|\sum_{j=1}^{p^t} a_{k+j}(f)\binom{p^t}{j}\right|_p \leqslant p^{-t} \max_{1\leqslant j\leqslant N} \left|\frac{a_{k+j}(f)}{j}\right|_p + \max_{N+1\leqslant j\leqslant p^t} |a_{k+j}(f)|_p$$

$$\leqslant p^{-t}N \max_{r\in J} |a_r(f)|_p + \Lambda(N).$$

From this inequality and (4.3) follows that

$$\max_{n\in J} |f(n+p^t)-f(n)|_p \leqslant p^{-t}N \max_{r\in J} |a_r(f)|_p + \Lambda(N).$$

Consequently,

$$\limsup_{t\to\infty} w_f(p^{-t}) \leqslant \Lambda(N)$$

and (2.3) follows from (4.2) since N can be chosen arbitrarily large.

Proof of Theorem 2. Sufficiency of (2.5). Suppose first that (2.5) holds, i.e., that $f \in \operatorname{Lip} \alpha$ on J, $0 < \alpha \leqslant 1$. Then f is bounded on J and, as in the proof of Theorem 1 we can assume, without loss of generality, that $|f(k)|_p \leqslant 1$ for $k \in J$. It follows then that $| a_n(f)_p \leqslant 1$ for every $n \in J$.

Also, since $f \in \operatorname{Lip} \alpha$ on J, we have

$$w_f(p^{-t}) = \max_{k\in J} |f(k+p^t)-f(k)|_p \leqslant M_f p^{-\alpha t} \quad \text{for} \quad t = 0, 1, 2,... \tag{4.4}$$

Then, by Lemma 3, we have, for every $n \geqslant 0$ and $t = 1, 2, r$ ...

$$|a_{n+p^t}(f)|_p \leqslant \max((1/p)|a_{n+j}(f)|_p, 1 \leqslant j \leqslant p^t - 1; M_f p^{-\alpha t}). \tag{4.5}$$

Since $|a_n(f)|_p \leqslant 1$ and $0 < \alpha \leqslant 1$, from (4.5) it follows that

$$|a_{n+p}(f)|_p \leqslant \max((1/p), M_f p^{-\alpha}) \leqslant \mu_f p^{-\alpha}$$

where $\mu_f = \max(1, M_f)$, or

$$|a_m|_p \leqslant \mu_f p^{-\alpha} \quad \text{for} \quad m \geqslant p. \tag{4.6}$$

Using (4.5) with $t = 2$ and $n \geqslant p$, and (4.6), we find that

$$\begin{aligned} | a_{n+p^2}(f)|_p &\leqslant \max(\mu_f p^{-\alpha-1}, M_f p^{-2\alpha}) \\ &\leqslant \mu_f \max(p^{-\alpha-1}, p^{-2\alpha}) \\ &\leqslant \mu_f p^{-2\alpha}. \end{aligned}$$

Hence,

$$| a_m |_p \leqslant \mu_f p^{-2\alpha} \qquad \text{for} \qquad m \geqslant p^2 + p.$$

Continuing this process we find by induction that

$$| a_m(f)|_p \leqslant \mu_f p^{-r\alpha} \qquad \text{for} \qquad m \geqslant p^r + \cdots + p. \tag{4.7}$$

Now it is easy to see that this inequality implies (2.4). Suppose that $n \geqslant p^2 + p$. Then, for some $s \geqslant 2$, we have

$$\begin{aligned} p^s + \cdots + p = p[(p^s - 1)/(p - 1)] &\leqslant n < p[(p^{s+1} - 1)/(p - 1)] \\ &= p^{s+1} + \cdots + p. \end{aligned}$$

Since $p^{s+2} \geqslant n(p - 1) + p \geqslant n$, we have, by (4.7),

$$| a_n(f)|_p \leqslant \mu_f p^{-s\alpha} \leqslant \mu_f p^{2\alpha} p^{-(s+2)\alpha} \leqslant \mu_f p^{2\alpha} n^{-\alpha}$$

for $n \geqslant p^2 + p$ and (2.4) follows.

Necessity of (2.5). Suppose next that (2.4) holds. We have then

$$| a_n(f)|_p \leqslant M n^{-\alpha} \qquad \text{for} \qquad n \geqslant 1 \tag{4.8}$$

where $0 < \alpha \leqslant 1$. As in the proof of the necessity part of Theorem 1, we have, by (4.3),

$$| f(n + p^t) - f(n)|_p \leqslant \max_{k \in J} \left| \sum_{j=1}^{p^t} a_{k+j}(f) \binom{p^t}{j} \right|_p$$

for every $n \geqslant 0$. Since

$$\sum_{j=1}^{p^t} a_{k+j}(f) \binom{p^t}{j} = p^t \sum_{j=1}^{p^t} \frac{a_{k+j}(f)}{j} \binom{p^t - 1}{j - 1}$$

it follows, by (4.8), that for every $k \in J$,

$$\begin{aligned} \left| \sum_{j=1}^{p^t} a_{k+j}(f) \binom{p^t}{j} \right|_p &\leqslant p^{-t} \max_{1 \leqslant j \leqslant p^t} \left| \frac{a_{k+j}(f)}{j} \right|_p \\ &\leqslant p^{-t} \max_{1 \leqslant j \leqslant p^t} \frac{j}{(k + j)^\alpha} (k + j)^\alpha \, | a_{k+j}(f)|_p \\ &\leqslant M p^{-t} \max_{1 \leqslant j \leqslant p^t} j^{1-\alpha} \\ &\leqslant M p^{-\alpha t}. \end{aligned}$$

Hence,

$$|f(n) + p^t) - f(n)|_p \leqslant Mp^{-\alpha t} \qquad \text{for} \qquad n \geqslant 0$$

and (2.5) follows.

Proof of Theorem 3. Suppose that $f(n + p^t) = f(n)$ for $n \in J$, with $t \geqslant 1$. The inequality (3.10) of Lemma 3 reduces then to

$$| a_{n+p^t}(f)|_p \leqslant \max((1/p) \mid a_{k+j} \mid_p , 1 \leqslant j \leqslant p^t - 1)$$

and it follows that for every $r = 1, 2,\ldots$ we have

$$n \geqslant rp^t \Rightarrow | a_n(f)|_p \leqslant Mp^{-r}$$

where $M = \max_{1 \leqslant j \leqslant p^t} | f(j)|_p$. If $n \geqslant p^t$ we have $rp^t \leqslant n < (r + 1) p^t$ for some $r \geqslant 1$ and so

$$| a_n(f)|_p \leqslant Mp^{-r} < Mp^{-n/p^t+1}$$

since $r > n/p^t - 1$. Let $M = p^m$. Then

$$| a_n(f)|_p < p^{m-n/p^t+1}.$$

Since the p-adic value is always an integral power of p and

$$-[n/p^t] < -n/p^t + 1,$$

we must have

$$| a_n(f)|_p \leqslant p^{m-[n/p^t]}$$

and Theorem 3 is proved.

Proof of Theorem 4. From (2.7) it follows that

$$w_f(p^{-t}) = \max_{k \in J} | f(k + p^t) - f(k)|_p \leqslant M\Omega(p^{-t}) \tag{4.9}$$

for $t = 1, 2,\ldots$ Since $\Omega(p^{-t}) \to 0$ $(t \to \infty)$, this inequality implies, by Lemma 1, that f is bounded on J. We can assume, without loss of generality, that $| f(k)|_p \leqslant 1$ for every $k \in J$. We have then $| a_n(f)|_p \leqslant 1$ for every $n \geqslant 0$ and consequently

$$\Lambda(n) = \max_{k \geqslant n} | a_k(f)|_p \geqslant 1 \qquad \text{for} \qquad n \geqslant 0. \tag{4.10}$$

Next, from

$$\liminf_{\lambda \to 0^+} \frac{\Omega(\lambda/p)}{\Omega(\lambda)} > 0 \tag{4.11}$$

it follows that we can find a constant $c > 0$ and an integer $T \geqslant 1$ such that

$$0 < \lambda \leqslant p^{-T} < 1 \Rightarrow \frac{\Omega(\lambda/p)}{\Omega(\lambda)} \geqslant c.$$

Let $s \geqslant 1$ be an integer such that

$$c \geqslant p^{-s} \qquad \text{and} \qquad M\Omega(p^{-T}) \geqslant p^{-s(s+1)}. \tag{4.12}$$

We have then, for $t \geqslant T$,

$$\Omega(p^{-t-1}) \geqslant c\Omega(p^{-t}) \geqslant p^{-s}\Omega(p^{-t})$$

and it follows, by induction, that, for every integer $\sigma \geqslant 1$,

$$\Omega(p^{-\sigma-1-t}) \geqslant p^{-s(\sigma+1)}\Omega(p^{-t}).$$

It follows, in particular, that

$$\Omega(p^{-s-1-t}) \geqslant p^{-s(s+1)}\Omega(p^{-t}). \tag{4.13}$$

Next, by Lemma 4 and (4.9) we have

$$\Lambda((1 + s(s + 1))\, p^t) \leqslant \max(p^{-s(s+1)}\Lambda(p^t), M\Omega(p^{-t})).$$

Since Λ is monotone decreasing and $p^{s+1} \geqslant 1 + s(s + 1)$, it follows that

$$\Lambda(p^{s+1+t}) \leqslant \Lambda((1 + s(s + 1))\, p^t) \leqslant \max(p^{-s(s+1)}\Lambda(p^t), M\Omega(p^{-t})). \tag{4.14}$$

In particular, if $t = T$, we have by (4.10) and (4.12),

$$\begin{aligned} \Lambda(p^{s+1+T}) &\leqslant \max(p^{-s(s+1)}\Lambda(p^T), M\Omega(p^{-T})) \\ &\leqslant \max(p^{-s(s+1)}, M\Omega(p^{-T})) \\ &\leqslant M\Omega(p^{-T}). \end{aligned} \tag{4.15}$$

Next, replacing t by $s + 1 + T$ in (4.14) and using inequalities (4.13) and (4.15), we find that

$$\begin{aligned} \Lambda(p^{2(s+1)+T}) &\leqslant \max(p^{-s(s+1)}\Lambda(p^{s+1+T}), M\Omega(p^{-s-1-T})) \\ &\leqslant \max(Mp^{-s(s+1)}\Omega(P^{-T}), M\Omega(p^{-s-1-T})) \\ &\leqslant M\Omega(p^{-s-1-T}). \end{aligned}$$

Continuing this argument, we find, by induction, that, for every $r \geqslant 1$,

$$\Lambda(p^{r(s+1)+T}) \leqslant M\Omega(p^{-(r-1)(s+1)-T}).$$

Using (4.13) we find easily that

$$\Omega(p^{-(r-1)(s+1)-T}) \leqslant p^{2s(s+1)}\Omega(p^{-(r+1)(s+1)+T}).$$

Hence,

$$\Lambda(p^{r(s+1)+T}) \leqslant Mp^{2s(s+1)}\Omega(p^{-(r+1)(s+1)-T}.$$

Let, finally, $n \geqslant p^{s+1+T}$. Then $p^{r(s+1)+T} \leqslant n < p^{(r+1)(s+1)+T}$ for some $r \geqslant 1$. By monotonicity of Λ and Ω we have

$$\Lambda(n) \leqslant \Lambda(p^{r(s+1)+T}) \leqslant Mp^{2s(s+1)}\Omega(p^{-(r+1)(s+1)-T})$$
$$\leqslant Mp^{2s(s+1)}\Omega(1/n)$$

and Theorem 4 is proved.

Proof of Theorem 5. We have, by hypothesis,

$$|a_n(f)|_p \leqslant M\Omega(1/n) \qquad \text{for} \qquad n \geqslant 1. \tag{4.16}$$

We have, by Lemma 2,

$$|f(n+p^t) - f(n)|_p \leqslant \max_{k \in J} \left| \sum_{j=1}^{p^t} a_{k+j}(f)\binom{p^t}{j}\right|_p \tag{4.17}$$

for every $n \in J$. Next, by (4.16),

$$\left| \sum_{j=1}^{p^t} a_{k+j}(f)\binom{p^t}{j}\right|_p = \left| p^t \sum_{j=1}^{p^t} (a_{k+j}(f)/j)\binom{p^t-1}{j-1}\right|_p$$
$$\leqslant p^{-t} \max_{1 \leqslant j \leqslant p^t} |a_{k+j}(f)/j|_p$$
$$\leqslant Mp^{-t} \max_{1 \leqslant j \leqslant p^t} j\Omega(1/k+j))$$
$$\leqslant Mp^{-t} \max_{1 \leqslant j \leqslant p^t} j\Omega(1/j).$$

From this inequality and (4.17) it follows that

$$w_f(p^{-t}) = \max_{n \in J} |f(n+p^t) - f(n)|_p \leqslant Mp^{-t} \max_{1 \leqslant j \leqslant p^t} j\Omega(1/j). \tag{4.18}$$

Since the function Ω satisfies condition (2.13), we have, by Lemma 5,

$$p^{-t} \max_{1 \leqslant j \leqslant p^t} j\Omega(1/j) = \mathcal{O}(\Omega(p^{-t})) \qquad (t \to \infty)$$

and Theorem 5 is proved.

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